Asymptotic behavior of guiding-center diffusion in a model of electrostatic turbulence

Michel De Leener
Physique Statistique et Plasmas, Code Postal 231, Université Libre de Bruxelles, B-1050 Brussels, Belgium
(Received 17 November 1993)

To compare with computer simulations of the diffusion of a test guiding center in a given electrostatic turbulence, a nonlinear theory is applied to the "randomly phased waves" model, with a single frequency ω and an arbitrary wave number spectrum. The asymptotic behavior of the diffusion coefficient D is determined in both limits of large and small turbulence amplitude a. For $a \to \infty$, the classical "frozen turbulence" scaling $D \propto a$ is found. For $a \to 0$, an unusual quadratic scaling is obtained: for all isotropic models, D goes to the same limit $(\sqrt{2}/\omega)a^2$. This behavior originates in the "two scales" character of this asymptotic problem. It is examined in detail on a simple form of the equation where the exact asymptotic solutions are obtained.

PACS number(s): 52.25.Fi, 52.65.+z, 05.40.+j, 05.45.+b

I. INTRODUCTION AND OUTLINE OF THE WORK

For more than two decades, a wealth of experimental and theoretical work has appeared concerning the "anomalous" diffusion in nuclear fusion plasmas [1]. This phenomenon is generally attributed to microscopic turbulence (with a length scale much smaller than the plasma radius). Among the various instabilities that may be at the origin of such turbulence, low-frequency electrostatic drift waves have drawn considerable attention; much theoretical work has been devoted to ion and electron diffusion near the instability and in strong turbulence.

The situation being particularly complex, some theorists have been interested in a relatively simple and reasonably realistic nonlinear model which has been called the two-dimensional electrostatic plasma of guiding centers [2]. This model corresponds to a strongly magnetized plasma where the energy density is so low that the external magnetic field is not modified locally; the electric field is then purely electrostatic, and the motion of the charges averaged over a period of their gyration is described by the relatively simple equation

$$\frac{d\mathbf{x}(t)}{dt} = c\,\frac{\mathbf{E}(\mathbf{x}(t),t)\times\mathbf{B}}{B^2}. \tag{1}$$

This model has been studied in different directions. The present work is concerned with the diffusion of a test guiding center in a *given* low-frequency random field $\mathbf{E}(\mathbf{x},t)$. In this spirit, following Taylor and McNamara [3], Montgomery [2], Dupree [4],... Misguich *et al.* [5] derived a nonlinear diffusion equation relating the mean squared displacement of a test charge moving in a stationary, homogeneous and isotropic field, to the correlation function of its Fourier components. For a discrete spectrum, this equation has the following form:

$$\frac{d^2\langle r^2(t)\rangle}{dt^2} = 2\left(\frac{c}{B}\right)^2 \sum_{\mathbf{k}} \langle \mathbf{E}_{\mathbf{k}}(t)\cdot\mathbf{E}_{-\mathbf{k}}(0)\rangle e^{-\frac{1}{4}k^2\langle r^2(t)\rangle}, \tag{2}$$

with the initial conditions $\langle r^2(0)\rangle = d\langle r^2(t)\rangle/dt|_{t=0} = 0$.

Thereafter, Pettini and co-workers [6,7] made computer simulations of test guiding-center diffusion in an electrostatic field given by the following particular form of the "randomly phased waves" model:

$$\mathbf{E}(\mathbf{x},t) = \frac{\widetilde{E}}{\sigma_S}\sum_{\mathbf{k}\in S} p(k)\,\mathbf{k}\cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \varphi_{\mathbf{k}}), \tag{3}$$

i.e., a weighted sum over a set S of harmonic waves, with a single frequency ω and wave numbers $\mathbf{k} = (2\pi/L)\,\mathbf{n}$ such that the field is periodic on the plane ($\mathbf{n}$ has integer components). [The choice of a fixed frequency instead of a dispersive function $\omega(\mathbf{k})$ was necessary to keep the computer time within reasonable limits.] The phases $\varphi_{\mathbf{k}}$ are uniform random numbers on $(0, 2\pi)$. The field amplitude $\widetilde{E}$ is defined as follows:

$$\widetilde{E}^2 = \langle|\mathbf{E}(\mathbf{x},t)|^2\rangle_{\mathbf{x},t} \tag{4}$$

(the average here is over one period $2\pi/\omega$ and over the $L\times L$ periodicity cell); it has been divided by

$$\sigma_S = \left(\frac{1}{2}\sum_{\mathbf{k}\in S} p^2(k)k^2\right)^{1/2}, \tag{5}$$

so that the average electrostatic energy density equals $(8\pi)^{-1}\widetilde{E}^2$.

The randomly phased waves model for $\mathbf{E}(\mathbf{x},t)$ was chosen because (a) the equations of motion (1) are equivalent to the following Hamiltonian form for $\mathbf{x} \equiv (x,y)$:

$$\frac{dx}{dt} = -\frac{\partial H}{\partial y},\quad \frac{dy}{dt} = \frac{\partial H}{\partial x}, \tag{6}$$

with $H = (c/B)\phi$, ϕ being the electrostatic potential; (b) it is known that, if the field defined in Eq. (3) involves more than two waves, these equations have chaotic solutions [8]. Since large-scale chaos corresponds to a fast diffusion process, this provides a mechanism for anomalous diffusion.

With the isotropic version of (3), the correlation function in Eq. (2) is readily obtained and one arrives at

$$\frac{d^2\langle r^2(t)\rangle}{dt^2} = a^2\cos(\omega t)\,\frac{1}{\sigma_S^2}\sum_{\mathbf{k}\in S} p^2(k)k^2 e^{-\frac{1}{4}k^2\langle r^2(t)\rangle}, \qquad (7)$$

where $a \equiv c\widetilde{E}/B$ is the "turbulence amplitude." We have integrated this equation numerically with various weights $p(k)$ and sets S such that $k_{\min} < |\mathbf{k}| < k_{\max}$; in all cases, the solution $\langle r^2(t)\rangle$ was observed to tend for $t \to \infty$ towards a diffusive asymptote:

$$\langle r^2(t)\rangle \underset{t\to\infty}{\sim} D(a)t, \quad \forall a. \qquad (8)$$

Typical examples of this general behavior are shown below (Figs. 2 and 3), displaying, in particular, the damped oscillations observed for small a [6]. These graphs correspond to the "one-wave diffusion" model [Eqs. (9) and (10)].

The differential equation (7) is the same as Eq. (32) in Ref. [7] where the weight $p(k)$ was set equal to k^{-3} as in the simulated models. However, although these models had a finite periodicity length L and were in fact anisotropic, the numerical results were compared to the more realistic theory for a continuous and isotropic wave number spectrum (Eq. (33) in Ref. [7]). For such a comparison to have any meaning, it was of course necessary to decide what relevant quantities should be identified or, equivalently, to specify a (model-dependent) system of time and length units. Since the models involve only one frequency, the natural time unit was the period $2\pi/\omega$. As for the length unit, it was argued that a reasonable choice was the largest wavelength $\lambda_{\max} = 2\pi/k_{\min}$: in Eq. (7), the smallest wave numbers have the largest weight and give the slowest decaying exponentials.

With these conventions, the theoretical diffusion coefficients obtained for $k_{\max} \gg k_{\min}$ are shown on Fig. 1 (solid line) as a function of a, on logarithmic scales; $\ln D(a)$ appears as a smooth curve, with two clear asymptotes: $D(a) \propto a$ at large amplitude and $D(a) \propto a^2$ for a small. We also show results from two simulations to indicate the general observation that the computer "experiments" yield diffusion coefficients which depend on the model and that there is only a rough agreement with the theoretical asymptotic predictions. The filled circles correspond to an anisotropic set S of vectors $\mathbf{k} = (2\pi/L)\,\mathbf{n}$ with $4 \le |\mathbf{n}| \le 48$ and $n_x \ge 0$ [7]; the open circles are results for the isotropic spectrum with $4 \le |\mathbf{n}| \le 48$ [9].

It was of course natural to examine whether the agreement improved when every computer experiment was compared to the theory corresponding strictly to the same model, i.e., to Eq. (7) modified to take anisotropy into account. This very simple generalization is described in Sec. II. In particular, we show that the Corssin-like [10] decoupling approximation, which is an essential ingredient of the theory for an arbitrary field [5], turns out to be exactly satisfied by the field defined in Eq. (3), under the conditions of Lagrangian stationarity and homogeneity.

Typical examples of the theoretical curves thus obtained for isotropic models are shown on Fig. 1. With the wave number spectra of both of the simulated models, the modified theory yielded results (crosses) very close to the continuous case. (Because of a symmetry inherent in the theory, it yields identical results for these two models: see Sec. II.) More generally, for large amplitudes, we observed that $D(a)$ always approached a linear asymptote, which only differed significantly from the $L \to \infty$ limit when the spectrum was very unrealistic: the largest difference (dashed line) corresponds to the one-wave diffusion model (see Sec. V). On the other hand, for small amplitudes, our results strongly suggested the existence of a quadratic asymptote with the *same* D/a^2 ratio for all models, as Fig. 1 indicates.

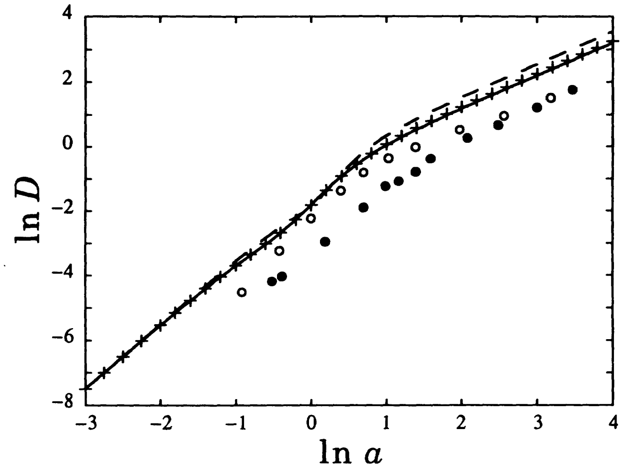

FIG. 1. Ln-ln plots of diffusion coefficient (D) vs turbulence amplitude (a) from theory and computer simulations (see text). Open and filled circles: simulation results for two models [7,9]. Solid line: continuous spectrum theory [7]. Crosses: discrete spectrum theory [Eq. (7)] for both models. Dashed line: one-wave diffusion model [Eq. (28)].

So, the theoretical equation (7), and the more general form it takes in anisotropic circumstances [Eq. (27)], appeared to have scaling properties which had to be clarified. The main result in this paper answers this question. The large-amplitude asymptotic problem led to the traditional "frozen turbulence" [11] scaling $D \propto a$ and our work there is merely a generalization to anisotropic situations. As for the small-amplitude limit, it turned out to be a nice problem in applied mathematics, leading to the result that, because the randomly phased waves field [Eq. (3)] involves a single frequency, there is a scaling $D \propto a^2$ which differs from the quasilinear one [12]; in particular, for all isotropic models, the asymptote does not depend on the wave number spectrum.

These problems appear most clearly on the particular form that the diffusion equation (7) takes when the spectrum is reduced to a single wave vector $\mathbf{k}_0$ (or rather the set of all wave vectors with the same length k_0). In terms of the following quantities

$$x = \omega t, \qquad \epsilon = \frac{k_0^2}{2\omega^2}a^2, \qquad f(x) = \frac{1}{4}k_0^2\langle r^2(t)\rangle, \qquad (9)$$

(7) then becomes

$$\frac{d^2 f(x)}{dx^2} = \epsilon \cos(x)\, e^{-f(x)}, \tag{10}$$

which we shall refer to as the one-wave diffusion equation. As a description of reality, this is of course a true caricature, but it displays very simply what is going on in general, as we show in detail in Sec. III.

To obtain from equations like (2) a qualitative expression for the diffusion coefficient, it has become traditional to approximate $\langle r^2(t)\rangle$ in the right-hand side by its asymptotic limit (Dt), which allows the equation to be integrated once and leads to a self-consistent relation for $D(a)$. Applying the corresponding procedure to Eq. (10), under the obvious condition that the "diffusion coefficient"

$$\Delta(\epsilon) \equiv \left.\frac{df(x)}{dx}\right|_{x\to\infty} \tag{11}$$

must be positive for the integral to converge, we obtain

$$\Delta = \epsilon \frac{\Delta}{1+\Delta^2}, \tag{12}$$

which yields a solution $\Delta = \sqrt{\epsilon - 1}$ *iff* $\epsilon > 1$.

Thus, when ϵ goes to infinity, Eq. (12) says that Δ behaves as $\sqrt{\epsilon}$. This turns out to be qualitatively correct (the exact result is $\sqrt{2\epsilon}$) and corresponds to the existence of an asymptotic limit

$$\epsilon \to \infty \quad \text{and} \quad x \to 0, \quad \text{with} \quad \sqrt{\epsilon}x = \xi \quad \text{finite}, \tag{13}$$

where $f(x,\epsilon) \sim \phi(\xi)$, so that

$$\Delta(\epsilon) \sim \sqrt{\epsilon} \left.\frac{d\phi(\xi)}{d\xi}\right|_{\xi\to\infty} = \sqrt{2\epsilon}. \tag{14}$$

This is the classical "frozen turbulence" [11] limit, which we examine in more detail in Sec. III A. In Sec. IV A, we show that this $\sqrt{\epsilon} \propto a$ asymptotic scaling applies to any wave number spectrum: from Eq. (7), we obtain

$$D(a) \underset{a\to\infty}{\sim} Ca \tag{15}$$

with

$$C = 4\left(\frac{\sum_{\mathbf{k}} p^2(k)}{\sum_{\mathbf{k}} p^2(k)k^2}\right)^{1/2}. \tag{16}$$

When the model is not isotropic, however, the expression that generalizes Eqs. (15) and (16) applies to the determinant of the diffusion matrix, not to the diffusion coefficient.

On the other hand, away from the $\epsilon \to \infty$ asymptote, Eq. (12) does not tell the truth. The dashed curve on Fig. 1 shows the numerical solution of Eq. (10). [With the units of Fig. 1, (9) leads to $\epsilon = (1/2)a^2$ and $\Delta = (\pi/2)D$.] Near $\epsilon = 1$, one observes a transition to a small-ϵ regime where Δ goes to zero linearly. It may seem surprising that the mere replacement of $f(x)$ by Δ times x has such misleading consequences. In fact, this approximation would work if the electrostatic field correlation in Eqs. (7) and (10), instead of behaving as a simple undamped oscillation, was going to zero with some *finite* time scale. Here, however, the scales of $\cos(x)$ and of $f(x)$ are both important.

So, for $\epsilon \to 0$, we are facing one of the asymptotic problems for which perturbation methods have been tailored in applied nonlinear mathematics [13]. As shown in an appendix, the "two-variable expansion" procedure can indeed be used here. However, the structure of (7) and (10) (and of the more general equation we introduce in Sec. II) is so simple that a formal analysis is not really necessary, as will be shown in Secs. III B and IV B. The essential result is that, for *any* isotropic model,

$$D(a) \underset{a\to 0}{\sim} \frac{\sqrt{2}}{\omega} a^2 \tag{17}$$

(in particular, for the one-wave diffusion equation, $\Delta(\epsilon) \sim \epsilon/\sqrt{2}$) while, as in the large-ϵ limit, the corresponding relation in the anisotropic case involves the determinant of the diffusion matrix.

At this point, it may be interesting to note that these large- and small-ϵ asymptotic results furnish a sounder alternative to the somewhat arbitrary choice of units introduced in Ref. [7] to allow a comparison of the theory with the computer simulations. Indeed, the asymptotic limits of D/a and D/a^2 have the dimensions of a length and a time, respectively, so that their explicit expressions (15) to (17) define units which make both asymptotes identical for all isotropic models, of course. This alternative definition does not lead to important numerical differences, however, as we show in Sec. V.

The following sections are organized as follows. Sec. II presents the derivation of the nonlinear equation generalizing (7) to the case where the randomly phased waves field [Eq. (3)] has an arbitrary spectrum [i.e., arbitrary S and $p(k)$]. The asymptotic limits of large and small amplitudes are then examined for the simple one-wave diffusion equation where all developments are transparent and the asymptotic equations can be solved analytically; this is the content of Secs. III A and III B. This analysis is then adapted to arbitrary spectra in Secs. IV A and IV B. In Sec. V, we present some numerical results and their theoretical implications. The Appendix gives a brief derivation of the results of Sec. III B with the two-variable expansion procedure.

II. DERIVATION OF THE NONLINEAR DIFFUSION EQUATION

Following Refs. [5] and [7], we now derive the general diffusion equation corresponding to the electrostatic field model defined in Eq. (3). Thus, we consider the two-dimensional motion of a test guiding center obeying Eq. (1), assuming that the random field $\mathbf{E}(\mathbf{x}(t), t)$ is stationary and homogeneous. Isotropy is not assumed, however, so that to describe the particle diffusion we need to consider the matrix $\mathbf{F}(t)$ defined as follows in terms of the displacement $\mathbf{r}(t) = \mathbf{x}(t) - \mathbf{x}(0)$:

$$F_{\alpha\beta}(t) = \langle r_\alpha(t) r_\beta(t)\rangle \equiv \langle [x_\alpha(t_0+t)-x_\alpha(t_0)][x_\beta(t_0+t)-x_\beta(t_0)]\rangle \quad (18)$$

with $(\alpha, \beta) = (x, y)$. We shall at some point consider the transformations of this matrix under finite rotations, but these questions will be rare and simple so that we shall allude to tensorial considerations only when necessary.

Very few ingredients are needed to obtain the diffusion equation.

(a) The stationarity assumed in (18) relates the time evolution of $\mathbf{F}(t)$ to the velocity autocorrelation function:

$$\frac{d^2}{dt^2}\mathbf{F}(t) = \langle \mathbf{v}(t)\mathbf{v}(0)\rangle + \langle \mathbf{v}(0)\mathbf{v}(t)\rangle. \quad (19)$$

(b) Since the guiding-center equation of motion (1) has the pleasant property that it relates the *velocity* to the field, we have

$$\langle \mathbf{v}(t)\mathbf{v}(0)\rangle = \left(\frac{c}{B}\right)^2 \langle (\mathbf{E}(\mathbf{x}(t),t)\times \mathbf{1}_z)(\mathbf{E}(\mathbf{x}(0),0)\times \mathbf{1}_z)\rangle. \quad (20)$$

(c) For an arbitrary field, we would proceed by expanding $\mathbf{E}(\mathbf{x},t)$ in a Fourier series and make the Corssin-like [10] approximation of replacing the average value by a product of averages, as follows:

$$\langle \mathbf{v}(t)\mathbf{v}(0)\rangle = \left(\frac{c}{B}\right)^2 \sum_{\mathbf{q}} \langle [\mathbf{E}_{\mathbf{q}}(t)\times\mathbf{1}_z][\mathbf{E}_{-\mathbf{q}}(0)\times\mathbf{1}_z] \times e^{i\mathbf{q}\cdot[\mathbf{x}(t)-\mathbf{x}(0)]}\rangle \quad (21)$$

$$\approx \left(\frac{c}{B}\right)^2 \sum_{\mathbf{q}} \langle [\mathbf{E}_{\mathbf{q}}(t)\times\mathbf{1}_z][\mathbf{E}_{-\mathbf{q}}(0)\times\mathbf{1}_z]\rangle \times \langle e^{i\mathbf{q}\cdot[\mathbf{x}(t)-\mathbf{x}(0)]}\rangle. \quad (22)$$

However, for the simple model field we are considering, this approximation is not needed; indeed, substituting its expression [Eq. (3)] in (20), we obtain

$$\langle \mathbf{v}(t)\mathbf{v}(0)\rangle = \left(c\frac{\widetilde{E}}{B}\right)^2 \frac{1}{\sigma_S^2} \sum_{\mathbf{k}',\mathbf{k}''\in S} p(k')p(k'')(\mathbf{k}'\times\mathbf{1}_z)(\mathbf{k}''\times\mathbf{1}_z) \times \langle \cos[\mathbf{k}'\cdot\mathbf{x}(t_0+t)-\omega(t_0+t)+\varphi_{\mathbf{k}'}] \times \cos[\mathbf{k}''\cdot\mathbf{x}(t_0)-\omega t_0+\varphi_{\mathbf{k}''}]\rangle. \quad (23)$$

When the product of cosines is then replaced by the corresponding sum, one observes that only one term can satisfy the homogeneity and stationarity requirements (with $\mathbf{k}' = \mathbf{k}''$) and that the phases then cancel; the average thus reduces to

$$\frac{1}{2}\,\delta_{\mathbf{k}'-\mathbf{k}''}\langle \cos[\mathbf{k}'\cdot\mathbf{r}(t) - \omega t]\rangle. \quad (24)$$

(d) Finally, to obtain a closed equation for $\mathbf{F}(t)$, we approximate this average by the first (Gaussian) term in its cumulant expansion:

$$\langle \cos[\mathbf{k}\cdot\mathbf{r}(t) - \omega t]\rangle \approx e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{F}(t)\cdot\mathbf{k}}\cos(\omega t), \quad (25)$$

where

$$\mathbf{k}\cdot\mathbf{F}(t)\cdot\mathbf{k} \equiv \sum_{\gamma\delta} k_\gamma F_{\gamma\delta}(t) k_\delta. \quad (26)$$

Substituting Eqs. (23)–(25) into (19), we arrive at the differential equation

$$\frac{d^2}{dt^2}\mathbf{F}(t) = a^2\cos(\omega t)\,\frac{1}{\sigma_S^2}\sum_{\mathbf{k}\in S} p^2(k)(\mathbf{k}\times\mathbf{1}_z)(\mathbf{k}\times\mathbf{1}_z) \times e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{F}(t)\cdot\mathbf{k}} \quad (27)$$

with the initial conditions: $\mathbf{F}(0) = d\mathbf{F}(t)/dt|_{t=0} = 0$.

The solution $\mathbf{F}(t)$ depends on the turbulence amplitude parameter $a \equiv c\widetilde{E}/B$ and on the spectrum [i.e., on S and $p(k)$]. However, the equation is bilinear in $\mathbf{k}$, so that its solution is invariant with respect to the change of sign of all the vectors $\mathbf{k}$ in S. Let us call $\overline{S}$ the set $\{\mathbf{k}\}$ thus obtained; then, Eq. (27) predicts the same diffusion law for test particles moving in fields corresponding to S, to $\overline{S}$ or to $(S+\overline{S})$. This is the reason why the models leading to the different simulation results [7,9] in Fig. 1 are described by the same theoretical curves. This particular symmetry of the theory clearly originates in the Gaussian approximation (25).

The differential equation describes the time evolution of the symmetric matrix $\mathbf{F}(t)$. To obtain any information (even a global one as the diffusion coefficient), we need in general to solve a system of three coupled equations. But clearly, any symmetry of S (pertaining to the symmetry group of the plane square lattice) will reflect itself in $\mathbf{F}(t)$, which may imply, for example, $F_{xy}(t) = 0$ and/or $F_{xx}(t) = F_{yy}(t)$ (in some reference frame). If both of these relations apply, one is left with the "isotropic" equation (7).

We have verified that the asymptotic property [Eq. (8)] of the diffusion equation (7) also holds for (27): for all spectra and all values of the amplitude, the derivative of the trace $\langle r^2(t)\rangle$ of $\mathbf{F}(t)$ for $t \to \infty$ goes to a constant, the diffusion coefficient D. We also observed that, as a function of a, D has clear asymptotic behaviors: $D \propto a$ for $a \to \infty$ and $D \propto a^2$ for $a \to 0$. These behaviors are the subject of the following sections. We start with the simple case of the one-wave diffusion equation.

III. ASYMPTOTIC BEHAVIOR OF THE ONE-WAVE DIFFUSION EQUATION

We now examine the asymptotic behavior of the differential equation

$$\frac{d^2 f(x,\epsilon)}{dx^2} = \epsilon\cos(x)\,e^{-f(x,\epsilon)}, \quad (28)$$

with $f(0,\epsilon) = df(x,\epsilon)/dx|_{x=0} = 0$, in the limits $\epsilon \to \infty$ and $\epsilon \to 0$.

A. Large amplitude limit

The existence of a classical asymptotic limit here is very easy to establish. Indeed, if we introduce in Eq. (28)

the variable $\xi = \sqrt{\epsilon}x$, we obtain

$$\frac{d^2\tilde{f}(\xi,\epsilon)}{d\xi^2} = \cos(\xi/\sqrt{\epsilon})\, e^{-\tilde{f}(\xi,\epsilon)}. \tag{29}$$

When we then take the limit defined in Eq. (13), the function $\cos(x = \xi/\sqrt{\epsilon})$ goes to 1 for any finite value of ξ, so that the asymptotic limit $\phi(\xi)$ of $f(x,\epsilon)$ obeys

$$\frac{d^2\phi(\xi)}{d\xi^2} = e^{-\phi(\xi)} \tag{30}$$

[with $\phi(0) = d\phi(\xi)/d\xi|_{\xi=0} = 0$]. This equation has a diffusive solution: starting from zero with a zero slope, the function $\phi(\xi)$ grows with a curvature that has its maximum at the origin and decreases continuously towards zero (exponentially for $\xi \to \infty$): $\phi(\xi)$ approaches an oblique asymptote.

In fact, the exact solution of (30) is known [14]:

$$\phi(\xi) = 2\ln\cosh\left(\frac{\xi}{\sqrt{2}}\right), \tag{31}$$

$$\frac{d\phi(\xi)}{d\xi} = \sqrt{2}\tanh\left(\frac{\xi}{\sqrt{2}}\right), \tag{32}$$

whence, the result we are looking for

$$\Delta(\epsilon) \equiv \left.\frac{df(x,\epsilon)}{dx}\right|_{x\to\infty} \underset{\epsilon\to\infty}{\sim} \sqrt{\epsilon}\left.\frac{d\phi(\xi)}{d\xi}\right|_{\xi\to\infty} = \sqrt{2\epsilon}. \tag{33}$$

Figure 2 shows how the numerical solution of Eq. (28)

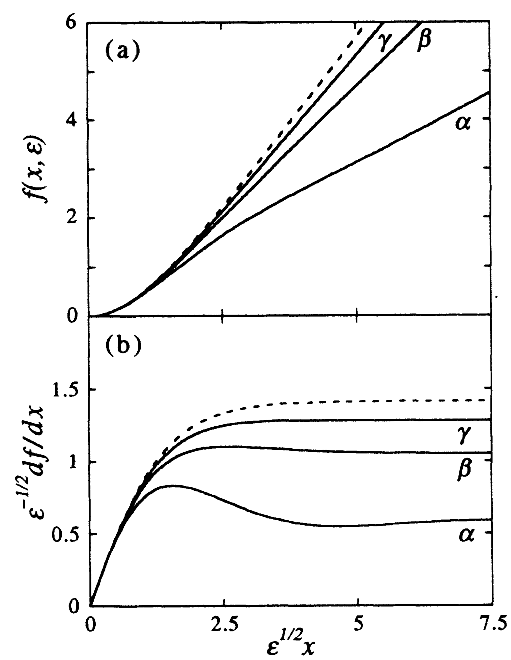

FIG. 2. One-wave diffusion model [Eq. (28)]. Approach towards the large-ϵ asymptotic limit of (a) the solution $f(x,\epsilon)$ and (b) its scaled derivative $\epsilon^{-1/2}df(x,\epsilon)/dx$. Numerical solutions (solid lines) are plotted vs the scaled variable $\epsilon^{-1/2}x$ for $\ln\epsilon = (\alpha)$ 0, (β) 1, (γ) 2. The dashed lines are the asymptotic limits [Eqs. (31) and (32)].

and its scaled derivative $(1/\sqrt{\epsilon})df/dx$ approach (31) and (32).

A very direct way to this result and to an estimate of the corrections is to multiply both sides of Eq. (28) by df/dx, which leads to

$$\begin{aligned}\frac{d}{dx}\left(\frac{df}{dx}\right)^2 &= -2\epsilon\cos(x)\frac{d}{dx}e^{-f}\\ &= -2\epsilon\frac{d}{dx}\left[\cos(x)\,e^{-f}\right] - 2\epsilon\sin(x)\,e^{-f}\end{aligned} \tag{34}$$

and, thus, after integrating from zero to infinity,

$$\left(\frac{df}{dx}\right)^2 \underset{x\to\infty}{\longrightarrow} 2\epsilon\left(1 - \int_0^\infty dx'\sin(x')e^{-f(x')}\right) \underset{\epsilon\to\infty}{\sim} 2\epsilon. \tag{35}$$

The integral yields an ϵ-independent correction, as can be seen by introducing the variable $\xi' = \sqrt{\epsilon}x'$ and expanding $\sin(x') \approx x' = \xi'/\sqrt{\epsilon}$.

B. Small amplitude limit

When ϵ goes to zero, the time scale of $f(x,\epsilon)$ goes to infinity. The oscillations then play an essential role in determining the solution of the one-wave diffusion equation (28). To see how the short- and long-time behavior combine in this solution, let us examine the straightforward expansion of $f(x,\epsilon)$ in powers of ϵ,

$$f(x,\epsilon) = f_0(x) + \epsilon f_1(x) + \epsilon^2 f_2(x) + \cdots . \tag{36}$$

When this is substituted into the differential equation and the resulting series is solved order by order, with due account of the initial conditions, one obtains for the first terms:

$$\begin{aligned} f_0(x) &= 0,\\ f_1(x) &= 1 - \cos(x),\\ f_2(x) &= -\frac{7}{8} + \frac{x^2}{4} + \cos(x) - \frac{1}{8}\cos(2x),\\ &\vdots \end{aligned} \tag{37}$$

One observes that (a) the nonlinearity of the differential equation generates, order by order, harmonics of the fundamental oscillation $\cos(x)$, (b) among the second-order contributions, there is a term $\propto (\epsilon x)^2$, which will ultimately dominate the others when x becomes large. This suggests that, when ϵ goes to zero, x will appear in two variables corresponding to different scales: the variable $x_0 = x$ of the oscillations, plus a second scaled variable $x_1 = \epsilon x$, which will govern the slow growth of the dominant terms (and, as can be seen in the few next orders, the behavior of the oscillation amplitudes).

When these two variables are substituted in the straightforward expansion, the resulting series can be rearranged to take the structure of a Fourier series:

$$f(x,\epsilon) = \psi(x_1,\epsilon) + \sum_n [c_n(x_1,\epsilon)\cos(nx_0) + s_n(x_1,\epsilon)\sin(nx_0)], \quad (38)$$

and the terms in the coefficients can be grouped according to the remaining powers of ϵ to yield, for example,

$$\psi(x_1,\epsilon) = \psi^{(0)}(x_1) + \epsilon\psi^{(1)}(x_1) + \epsilon^2\psi^{(2)}(x_1) + \cdots . \quad (39)$$

A closer examination of Eq. (37) and the few next orders in (36) allows us to see that (38) has the following relatively simple form:

$$f(x,\epsilon) = \psi^{(0)}(x_1) + \epsilon\,[\psi^{(1)}(x_1) + c_1^{(1)}\cos(x_0)] + \epsilon^2\,[\psi^{(2)}(x_1) + c_1^{(2)}(x_1)\cos(x_0) + c_2^{(2)}(x_1)\cos(2x_0) + s_1^{(2)}(x_1)\sin(x_0)] + O(\epsilon^3). \quad (40)$$

Of course, the coefficients in this series are only obtained as their series expansions in powers of x_1, arising from the small-ϵ and finite-x expansion (36) and (37); for example,

$$\psi^{(0)}(x_1) = \frac{x_1^2}{4} - \frac{x_1^4}{48} + \cdots, \qquad c_1^{(1)} = -1 + \frac{x_1^2}{4} + \cdots . \quad (41)$$

However, to determine these unknown functions (and to confirm that an asymptotic solution of this form does exist), it suffices to substitute Eq. (40) into (28). The left-hand side becomes

$$\frac{d^2 f(x,\epsilon)}{dx^2} = -\,\epsilon\, c_1^{(1)}(x_1)\cos(x_0) + \epsilon^2\left[\frac{d^2\psi^{(0)}(x_1)}{dx_1^2} - \frac{dc_1^{(1)}(x_1)}{dx_1}\sin(x_0) - c_1^{(2)}(x_1)\cos(x_0) - 4\,c_2^{(2)}(x_1)\cos(2x_0) - s_1^{(2)}(x_1)\sin(x_0)\right] + O(\epsilon^3). \quad (42)$$

On the right-hand side, expanding the exponential leads to

$$\epsilon\cos(x_0)\,e^{-\psi^{(0)}(x_1)}\{1 - \epsilon\,[\psi^{(1)}(x_1) + c_1^{(1)}\cos(x_0)] + O(\epsilon^2)\}$$
$$= \epsilon\cos(x_0)\,e^{-\psi^{(0)}(x_1)} - \epsilon^2 e^{-\psi^{(0)}(x_1)}\left[\psi^{(1)}(x_1)\cos(x_0) + c_1^{(1)}(x_1)\frac{1}{2}[1+\cos(2x_0)]\right] + O(\epsilon^3). \quad (43)$$

Equating first-order terms, we find

$$c_1^{(1)}(x_1) = -\,e^{-\psi^{(0)}(x_1)}, \quad (44)$$

so that the formal solution (40) becomes

$$f(x,\epsilon) = \psi^{(0)}(x_1) + \epsilon[\psi^{(1)}(x_1) - e^{-\psi^{(0)}(x_1)}\cos(x_0)] + O(\epsilon^2). \quad (45)$$

To determine $\psi^{(0)}(x_1)$, we simply collect the terms of second order that do not depend on x_0; we thus obtain the differential equation

$$\frac{d^2\psi^{(0)}(x_1)}{dx_1^2} = -\frac{1}{2}\,c_1^{(1)}(x_1) = \frac{1}{2}\,e^{-2\psi^{(0)}(x_1)} \quad (46)$$

[with $\psi^{(0)}(0) = d\psi^{(0)}(x_1)/dx_1|_{x_1=0} = 0$]. This equation is the same as (30), whence the solution

$$\psi^{(0)}(x_1) = \ln\cosh\left(\frac{x_1}{\sqrt{2}}\right). \quad (47)$$

Thus, the small- and large-ϵ asymptotic limits of $f(x,\epsilon)$ are the same functions. However, the variables being different, so are the derivatives: instead of Eq. (32), (45) gives

$$\frac{df(x,\epsilon)}{dx} = \epsilon\left[\frac{d\psi^{(0)}(x_1)}{dx_1} + e^{-\psi^{(0)}(x_1)}\sin(x_0)\right] + O(\epsilon^2)$$
$$\equiv \epsilon\left[\frac{1}{\sqrt{2}}\tanh(\epsilon x/\sqrt{2}) + \frac{\sin(x)}{\cosh(\epsilon x/\sqrt{2})}\right] + O(\epsilon^2). \quad (48)$$

Figure 3 shows how the numerical solution of Eq. (28) approaches its asymptotic limit when $\epsilon \to 0$. On Fig. 3(a), $f(x,\epsilon)$ does look like the form we have arrived at [Eq. (45)], i.e., the sum of a growing function of ϵx and of an oscillation that starts like $\epsilon[1-\cos(x)]$ [Eq. (37)] and decays when the growing part comes close to its oblique asymptote. On Fig. 3(b), the scaled derivative $(1/\epsilon)df(x,\epsilon)/dx$ is plotted versus ϵx. The second term in the asymptotic limit [Eq. (48)] has a typical two-variable form and raises the usual problem of the strict $\epsilon \to 0$ limit: when expressed in terms of the scaled variable, the oscillation $\sin[\epsilon^{-1}(\epsilon x)]$ has a frequency that goes to infinity. In the figure, we therefore plotted the three sets of values of the asymptotic form where $\sin(x)$ equals -1, 0, and $+1$, and we compared them with *some* of the corresponding values from the numerical solution for $\ln\epsilon = -4$. The complete solution is shown for $\ln\epsilon = -2$.

In accordance with Eq. (8), the amplitude of the awkward oscillation in Eq. (48) goes to zero when we let x go to infinity (with ϵ fixed), so that the "diffusion coefficient" $\Delta(\epsilon)$ [Eq. (11)] is well behaved in the $\epsilon \to 0$ limit, and we thus find

$$\Delta(\epsilon) \equiv \left.\frac{df(x,\epsilon)}{dx}\right|_{x\to\infty} \underset{\epsilon\to 0}{\frown} \epsilon\left.\frac{d\psi^{(0)}(x_1)}{dx_1}\right|_{x_1\to\infty} = \frac{\epsilon}{\sqrt{2}}. \quad (49)$$

This is the result we wanted to establish: when ϵ goes to zero, the diffusion coefficient $\Delta(\epsilon)$ vanishes linearly.

This asymptotic solution of the one-wave diffusion equation can also be arrived at with the multiple-variable expansion method [13]; this derivation is given in an ap-

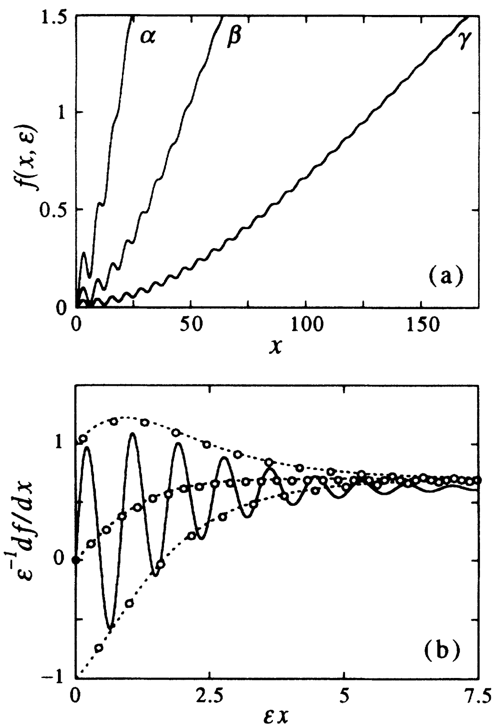

FIG. 3. One-wave diffusion model [Eq. (28)]. Approach towards the small-ϵ asymptotic limit. (a) Plot of the numerical solution $f(x,\epsilon)$ vs x for $\ln\epsilon = (\alpha)\ -2,\ (\beta)\ -3,\ (\gamma)\ -4$. (b) Plot of the scaled derivative $\epsilon^{-1}df(x,\epsilon)/dx$ vs the scaled variable ϵx. Solid line: numerical solution for $\ln\epsilon = -2$. Dotted lines: three sets of values from the asymptotic solution [Eq. (48)] for $\sin(x)$ equal to -1, 0, and $+1$. Open circles: for comparison, some of the corresponding points from the numerical solution for $\ln\epsilon = -4$.

pendix. Let us note however that this method has been initially devised to deal with the effect of small perturbations on the known solutions of differential equations. The problem of Eq. (28) is different: for $\epsilon = 0$, its solution reduces to $f(x,0) = 0$. So, when ϵ grows from zero, the structure of the asymptotic solution is determined by that of the "perturbation" in the right-hand side of the equation. This explains why this Fourier series structure appears so clearly in the pedestrian derivation above.

IV. ASYMPTOTIC BEHAVIOR OF THE GENERAL DIFFUSION EQUATION

We here address the large- and small-amplitude asymptotic problems associated with the diffusion equation (27), where the wave number spectrum is arbitrary. The solutions of these problems are very similar to those we obtained in Sec. III for the one-wave diffusion equation (28). This appears clearly when (27) is rewritten as

$$\frac{d^2}{dt^2}\mathbf{F}(t,\eta) = \eta\cos(\omega t)\,\widetilde{\mathbf{K}}(\mathbf{F}(t,\eta)), \qquad (50)$$

where we introduced the dimensionless perturbation parameter η (associated with a^2) and where

$$\widetilde{\mathbf{K}}(\mathbf{F}) \equiv \frac{a^2}{\sigma_S^2}\sum_{\mathbf{k}\in S} p^2(k)\,\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}\,e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{F}\cdot\mathbf{k}} \qquad (51)$$

only depends on t through the unknown $\mathbf{F}(t)$. The notation $\widetilde{\mathbf{k}}$ for $\mathbf{k}\times\mathbf{1}_z$ indicates that this vector is obtained by rotating $\mathbf{k}$ through an angle of $\pi/2$ so that $\mathbf{k} \equiv (k_x, k_y)$ becomes $\widetilde{\mathbf{k}} \equiv (k_y, -k_x)$. The same notation is used for the matrix $\widetilde{\mathbf{K}}$, with the same meaning. This finite rotation is the only transformation we need to consider. For the sake of brevity, we shall also use, instead of (51), the more compact notation

$$\widetilde{\mathbf{K}}(\mathbf{F}) \equiv \{\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}\,e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{F}\cdot\mathbf{k}}\}_{\mathbf{k}}. \qquad (52)$$

Since $\mathbf{r}(t)$ and $\mathbf{k}$ are vectors, the matrices $\mathbf{F}$ and $\widetilde{\mathbf{K}}$, defined in Eqs. (18) and (51), correspond to tensors: they are positively weighted sums (proportional to average values) of the dyads $\mathbf{r}(t)\mathbf{r}(t)$ and $\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}$, and they have the same symmetry properties. Only the following ones will be needed; if we denote by $\mathbf{A}$ the matrix associated with the dyad $\mathbf{a}\,\mathbf{a}$, then

(a) $\mathbf{A}$ obeys the following transformation law under a rotation of $\pi/2$:

$$\mathbf{A} = \begin{pmatrix} a_x^2 & a_x a_y \\ a_y a_x & a_y^2 \end{pmatrix} \longrightarrow \widetilde{\mathbf{A}} = \begin{pmatrix} a_y^2 & -a_y a_x \\ -a_x a_y & a_x^2 \end{pmatrix}. \qquad (53)$$

(b) The trace and the determinant of $\mathbf{A}$ are positive semidefinite; indeed,

$$\mathbf{q}\cdot\mathbf{A}\cdot\mathbf{q} \equiv \left(\sum_\alpha q_\alpha a_\alpha\right)^2 \equiv (\mathbf{q}\cdot\mathbf{a})^2 \geq 0, \qquad \forall\mathbf{q}, \qquad (54)$$

and therefore

$$\mathrm{tr}\mathbf{A} \geq 0 \qquad \text{and} \qquad \det\mathbf{A} \geq 0. \qquad (55)$$

(c) If $\mathbf{A}$ depends on a variable z, it is easy to see that

$$\widetilde{\mathbf{A}}(z):\frac{d}{dz}\mathbf{A}(z) \equiv \sum_{\alpha,\beta}\widetilde{A}_{\alpha\beta}(z)\frac{d}{dz}A_{\alpha\beta}(z) = \frac{d}{dz}\det\mathbf{A}(z). \qquad (56)$$

In addition, we shall make use of the invariance of the scalar quantity

$$\mathbf{q}\cdot\mathbf{A}\cdot\mathbf{q} \equiv \widetilde{\mathbf{q}}\cdot\widetilde{\mathbf{A}}\cdot\widetilde{\mathbf{q}}, \qquad (57)$$

which implies, in particular, the two following identities:

$$\widetilde{\mathbf{q}}\widetilde{\mathbf{q}}\,e^{-\frac{1}{2}\mathbf{q}\cdot\mathbf{A}\cdot\mathbf{q}} = -2\frac{\partial}{\partial\widetilde{\mathbf{A}}}\,e^{-\frac{1}{2}\mathbf{q}\cdot\mathbf{A}\cdot\mathbf{q}}, \qquad (58)$$

$$\widetilde{\mathbf{K}}(\mathbf{F}) \equiv -2\frac{\partial}{\partial\widetilde{\mathbf{F}}}\{e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{F}\cdot\mathbf{k}}\}, \qquad (59)$$

the latter being a direct consequence of Eq. (52).

A. Large amplitude limit

As for Eq. (28), there exists for (50) a limit where $\eta \to \infty$ and $t \to 0$, $\tau \equiv \sqrt{\eta}\, t$ remaining finite. Then, $\cos(\omega t) \to 1$, $\mathbf{F}(t,\eta) \sim \mathbf{\Phi}(\tau)$ and the differential equation thus has the following limit:

$$\frac{d^2}{d\tau^2}\mathbf{\Phi}(\tau) = \widetilde{\mathbf{K}}(\mathbf{\Phi}(\tau)) \qquad (60)$$

[with $\mathbf{\Phi}(0) = d\mathbf{\Phi}(\tau)/d\tau|_{\tau=0} = 0$].

For any model [i.e., any domain S and weight $p(k)$ in Eq. (51)], (60) defines $\mathbf{\Phi}(\tau)$, the large-amplitude limit of $\mathbf{F}(t,\eta) \equiv \langle \mathbf{r}(t,\eta)\mathbf{r}(t,\eta)\rangle$, from which the diffusion matrix $\mathbf{D}(\eta \to \infty)$ can be obtained as the limit of its derivative when τ goes to infinity. However, similarly to what we showed in Eqs. (34) and (35), an exact information concerning the diffusion matrix can be derived directly as a first integral of the differential equation. Indeed, with the help of Eqs. (56) and (59), (60) leads to

$$\begin{aligned}\frac{d}{d\tau}\left(\det\frac{d\mathbf{\Phi}}{d\tau}\right) &= \frac{d\widetilde{\mathbf{\Phi}}}{d\tau} : \frac{d^2\mathbf{\Phi}}{d\tau^2} \\ &= -2\frac{d\widetilde{\mathbf{\Phi}}}{d\tau} : \frac{\partial}{\partial\widetilde{\mathbf{\Phi}}}\{e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{\Phi}\cdot\mathbf{k}}\}_{\mathbf{k}} \\ &= -2\frac{d}{d\tau}\{e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{\Phi}\cdot\mathbf{k}}\}_{\mathbf{k}}, \end{aligned}\qquad (61)$$

whence the integral

$$\det\frac{d\mathbf{\Phi}}{d\tau} = 2\{1 - e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{\Phi}\cdot\mathbf{k}}\}_{\mathbf{k}}. \qquad (62)$$

Now, what we wrote about $\phi(\xi)$ after Eq. (30) is easily translated to hold here for $\mathbf{k}\cdot\mathbf{\Phi}(\tau)\cdot\mathbf{k}$ ($\forall\mathbf{k}$), and thus for the determinant and the trace of $\mathbf{\Phi}(\tau)$. Indeed, from Eq. (54), we know that these quantities are positive semidefinite:

$$\mathbf{k}\cdot\mathbf{\Phi}(\tau)\cdot\mathbf{k} \equiv \langle(\mathbf{k}\cdot\mathbf{r}(\tau))^2\rangle \geq 0. \qquad (63)$$

They start from zero at $\tau = 0$ with a zero slope and we now show that they grow with a curvature that is also positive semidefinite: from Eq. (60) again,

$$\begin{aligned}\mathbf{k}\cdot\frac{d^2\mathbf{\Phi}(\tau)}{d\tau^2}\cdot\mathbf{k} &= \mathbf{k}\cdot\{\widetilde{\mathbf{k}}'\widetilde{\mathbf{k}}' e^{-\frac{1}{2}\mathbf{k}'\cdot\mathbf{\Phi}\cdot\mathbf{k}'}\}_{\mathbf{k}'}\cdot\mathbf{k} \\ &\equiv \{(\mathbf{k}\cdot\widetilde{\mathbf{k}}')^2 e^{-\frac{1}{2}\mathbf{k}'\cdot\mathbf{\Phi}\cdot\mathbf{k}'}\}_{\mathbf{k}'} \geq 0, \end{aligned}\qquad (64)$$

the equality holding only when all exponents $\mathbf{k}'\cdot\mathbf{\Phi}(\tau)\cdot\mathbf{k}'$ go to infinity along an oblique asymptote. Therefore,

$$\det\frac{d\mathbf{\Phi}}{d\tau} \underset{\tau\to\infty}{\sim} 2\{1\}_{\mathbf{k}} = 2\,\frac{a^2}{\sigma_S^2}\sum_{\mathbf{k}\in S} p^2(k). \qquad (65)$$

Thus, when the amplitude a goes to infinity, the asymptotic limit of the diffusion matrix $\mathbf{D} \equiv d\mathbf{\Phi}/d\tau|_{\tau\to\infty}$ obeys the following relation:

$$\frac{1}{a^2}\det\mathbf{D} = 4\left(\frac{\sum_{\mathbf{k}\in S} p^2(k)}{\sum_{\mathbf{k}\in S} p^2(k)k^2}\right), \qquad (66)$$

which expresses the linear "frozen turbulence" [11] scaling behavior of the diffusion equation (27) in the large-amplitude limit. If the spectrum is isotropic, Eq. (66) leads to the announced result [Eqs. (15) and (16)] for the diffusion coefficient D.

B. Small amplitude limit

From the formal similarity between Eq. (50) and the one-wave diffusion equation (28), it follows that the analysis leading to the small-ϵ limit, in Sec. III B and in the Appendix, can be easily adapted here: the only important modification is that the functional dependence $\exp(-f(x,\epsilon))$ is replaced by $\widetilde{\mathbf{K}}(\mathbf{F}(t,\eta))$, defined in Eq. (51) as a linear combination of exponentials of $-\frac{1}{2}\mathbf{k}\cdot\mathbf{F}(t)\cdot\mathbf{k}$. Thus, when we write $\mathbf{F}$ as a two-variable expansion in terms of $t_0 = t$ and $t_1 = \eta t$

$$\begin{aligned}\mathbf{F}(t) &= \mathbf{F}^{(0)}(t_0,t_1) + \eta\,\mathbf{F}^{(1)}(t_0,t_1) \\ &\quad + \eta^2\,\mathbf{F}^{(2)}(t_0,t_1) + \cdots, \end{aligned}\qquad (67)$$

and expand $\widetilde{\mathbf{K}}(\mathbf{F})$ as

$$\widetilde{\mathbf{K}}(\mathbf{F}) = \widetilde{\mathbf{K}}(\mathbf{F}^{(0)}) + \eta\left[\mathbf{F}^{(1)} : \frac{\partial}{\partial\mathbf{F}^{(0)}}\right]\widetilde{\mathbf{K}}(\mathbf{F}^{(0)}) + \cdots, \qquad (68)$$

the analysis follows the same path and one obtains [compare with Eqs. (45) and (46)]

$$\begin{aligned}\mathbf{F}(t) = \mathbf{\Psi}^{(0)}(t_1) + \eta\Big(&\mathbf{\Psi}^{(1)}(t_1) \\ &- \frac{1}{\omega^2}\widetilde{\mathbf{K}}(\mathbf{\Psi}^{(0)}(t_1))\cos(\omega t_0)\Big) + O(\eta^2) \end{aligned}\qquad (69)$$

and

$$\frac{d^2\mathbf{\Psi}^{(0)}}{dt_1^2} = -\frac{1}{2\omega^2}\left[\widetilde{\mathbf{K}}(\mathbf{\Psi}^{(0)}) : \frac{\partial}{\partial\mathbf{\Psi}^{(0)}}\right]\widetilde{\mathbf{K}}(\mathbf{\Psi}^{(0)}) \qquad (70)$$

[with $\mathbf{\Psi}^{(0)}(0) = d\mathbf{\Psi}^{(0)}(t_1)/dt_1|_{t_1=0} = 0$].

Here as in the large-amplitude limit, solving these equations for any particular spectrum yields the small-amplitude asymptotic limit of $\mathbf{F}(t)$ and its first derivative, and thus allows us to evaluate the diffusion matrix $\mathbf{D}(\eta \to 0)$. However, as in the large-amplitude limit again, a first integral of Eq. (70) can be obtained and leads for $t_1 \to \infty$ to an exact expression of the determinant of $d\mathbf{\Psi}^{(0)}(t_1)/dt_1$ (and hence of $\mathbf{D}$) in terms of the parameters defining the model. Using Eq. (56), we have indeed

$$\begin{aligned}\frac{d}{dt_1}\det\frac{d\mathbf{\Psi}^{(0)}}{dt_1} &= \frac{d\widetilde{\mathbf{\Psi}}^{(0)}}{dt_1} : \frac{d^2\mathbf{\Psi}^{(0)}}{dt_1^2} \\ &\equiv -\frac{1}{2\omega^2}\sum_{\alpha,\beta}\frac{d\widetilde{\psi}^{(0)}_{\alpha\beta}}{dt_1}\sum_{\gamma,\delta}\widetilde{K}_{\gamma\delta}(\mathbf{\Psi}^{(0)}) \\ &\qquad\times\frac{\partial}{\partial\psi^{(0)}_{\gamma\delta}}\widetilde{K}_{\alpha\beta}(\mathbf{\Psi}^{(0)}). \end{aligned}\qquad (71)$$

Now, by Eq. (59),

$$\frac{\partial}{\partial\psi^{(0)}_{\gamma\delta}}\widetilde{K}_{\alpha\beta}(\mathbf{\Psi}^{(0)}) = -2\frac{\partial^2}{\partial\psi^{(0)}_{\gamma\delta}\partial\widetilde{\psi}^{(0)}_{\alpha\beta}}\{e^{-\frac{1}{2}\mathbf{k}\cdot\mathbf{\Psi}^{(0)}\cdot\mathbf{k}}\}_{\mathbf{k}} = \frac{\partial}{\partial\widetilde{\psi}^{(0)}_{\alpha\beta}}K_{\gamma\delta}(\mathbf{\Psi}^{(0)}), \tag{72}$$

so that, using (56) again,

$$\frac{d}{dt_1}\det\frac{d\mathbf{\Psi}^{(0)}}{dt_1} = -\frac{1}{2\omega^2}\sum_{\alpha,\beta}\frac{d\widetilde{\psi}^{(0)}_{\alpha\beta}}{dt_1}\sum_{\gamma,\delta}\widetilde{K}_{\gamma\delta}(\mathbf{\Psi}^{(0)})\frac{\partial}{\partial\widetilde{\psi}^{(0)}_{\alpha\beta}}K_{\gamma\delta}(\mathbf{\Psi}^{(0)}) = -\frac{1}{2\omega^2}\sum_{\alpha,\beta}\frac{d\widetilde{\psi}^{(0)}_{\alpha\beta}}{dt_1}\frac{\partial}{\partial\widetilde{\psi}^{(0)}_{\alpha\beta}}\det\mathbf{K}(\mathbf{\Psi}^{(0)}) = -\frac{1}{2\omega^2}\frac{d}{dt_1}\det\mathbf{K}(\mathbf{\Psi}^{(0)}) \tag{73}$$

and finally

$$\det\frac{d\mathbf{\Psi}^{(0)}}{dt_1} = \frac{1}{2\omega^2}[\det\mathbf{K}(\mathbf{\Psi}^{(0)}(0)) - \det\mathbf{K}(\mathbf{\Psi}^{(0)}(t_1))]. \tag{74}$$

The second term vanishes when t_1 goes to infinity. More precisely, in the definition of $\mathbf{K}(\mathbf{\Psi}^{(0)})$ [Eqs. (51) and (52)], the scalar $\mathbf{k}\cdot\mathbf{\Psi}^{(0)}(t_1)\cdot\mathbf{k}$ in the exponential is positive semidefinite [Eqs. (54) and (55)] and, as we now show, has a positive semidefinite curvature (the same holding for the trace and the determinant of $\mathbf{\Psi}^{(0)}$):

$$\mathbf{k}\cdot\frac{d^2\mathbf{\Psi}^{(0)}}{dt_1^2}\cdot\mathbf{k} = -\frac{1}{2\omega^2}\left[\widetilde{\mathbf{K}}(\mathbf{\Psi}^{(0)}):\frac{\partial}{\partial\mathbf{\Psi}^{(0)}}\right]\mathbf{k}\cdot\widetilde{\mathbf{K}}(\mathbf{\Psi}^{(0)})\cdot\mathbf{k} \equiv \frac{1}{4\omega^2}\{\{(\widetilde{\mathbf{k}}'\cdot\widetilde{\mathbf{k}}'')^2(\mathbf{k}\cdot\mathbf{k}'')^2 e^{-\frac{1}{2}\mathbf{k}'\cdot\mathbf{\Psi}^{(0)}\cdot\mathbf{k}'} \times e^{-\frac{1}{2}\mathbf{k}''\cdot\mathbf{\Psi}^{(0)}\cdot\mathbf{k}''}\}_{\mathbf{k}'}\}_{\mathbf{k}''} \geq 0. \tag{75}$$

So, $\mathbf{k}\cdot\mathbf{\Psi}^{(0)}(t_1)\cdot\mathbf{k}$ increases monotonously with t_1 and approaches an oblique asymptote in the limit $t_1 \to \infty$: then, all components of $\mathbf{K}(\mathbf{\Psi}^{(0)})$ go (exponentially) to zero and

$$\det\frac{d\mathbf{\Psi}^{(0)}}{dt_1} \underset{t_1\to\infty}{\sim} \frac{1}{2\omega^2}\det\mathbf{K}(\mathbf{\Psi}^{(0)}(0)) \equiv \frac{1}{2\omega^2}\left(\frac{a^2}{\sigma_S^2}\right)^2\det\left(\sum_{\mathbf{k}\in S}p^2(k)\,\mathbf{k}\mathbf{k}\right). \tag{76}$$

This result yields an explicit expression of the diffusion matrix $\mathbf{D}(\eta\to 0)$. From Eq. (69), we have indeed

$$\frac{d\mathbf{F}(t)}{dt} = \eta\left[\frac{d\mathbf{\Psi}^{(0)}(t_1)}{dt_1} + \frac{1}{\omega}\widetilde{\mathbf{K}}(\mathbf{\Psi}^{(0)}(t_1))\sin(\omega t_0)\right] + O(\eta^2) \tag{77}$$

[compare with Eq. (48)] and since $\mathbf{K}(\mathbf{\Psi}^0)$ vanishes at infinity,

$$\mathbf{D}(\eta) \equiv \left.\frac{d\mathbf{F}(t,\eta)}{dt}\right|_{t\to\infty} \underset{\eta\to\infty}{\sim} \eta\left.\frac{d\mathbf{\Psi}^{(0)}(t_1)}{dt_1}\right|_{t_1\to\infty}. \tag{78}$$

With Eq. (76), this leads to our main result: the diffusion matrix $\mathbf{D}(a\to 0)$ obeys the asymptotic relation

$$\frac{1}{a^4}\det\mathbf{D} = \frac{2}{\omega^2}\det\left(\frac{\sum_{\mathbf{k}\in S}p^2(k)\,\mathbf{k}\mathbf{k}}{\sum_{\mathbf{k}\in S}p^2(k)k^2}\right). \tag{79}$$

This agrees with our numerical observations; in particular, (a) for all models, the diffusion coefficient vanishes as a^2, (b) for isotropic models, the ratio D/a^2 is *independent* of the wave number spectrum: Eq. (79) becomes (17). Let us stress again that this property is related to the presence of a single frequency in the version of the model electrostatic field that we have considered [Eq. (3)].

V. CONCLUSIONS

To allow a more precise comparison of a nonlinear theory [5] with computer simulations [7,9] of the diffusion of a test guiding center in a given electrostatic turbulence, we have derived the theoretical equations corresponding to the randomly phased waves model [Eq. (3)] for the electrostatic field, with a single frequency and an arbitrary wave number spectrum. It appeared that the only approximation that was needed was (25), i.e., that the diffusion corresponds to a Gaussian process. So, if the precision of the numerical simulations can be well assessed, the discrepancy has to be related to this approximation. This is of course no surprise: with the field (3), the Hamiltonian equations of motion [Eq. (6)] generate chaotic trajectories, and it is known that a Gaussian description of these is only approximate (and inadequate when the amplitude is small) [7]. We have examined the asymptotic behavior of the diffusion coefficient when the turbulence amplitude $a \equiv c\widetilde{E}/B$ goes to infinity or to zero. The answer to these questions was first obtained in the particular case of the one-wave diffusion equation (28), where the asymptotic equations can be solved exactly. The general results are given in Eqs. (66) and (79): the diffusion matrix $\mathbf{D}(a)$ scales as a in the limit $a\to\infty$ and as a^2 for a going to zero. If the field is isotropic, we obtain explicit expressions for the limits of the ratios D/a or D/a^2 [Eqs. (15)–(17)].

The small-a result is very striking: the limit of D/a^2 does not depend on the wave number spectrum. As we stressed throughout, this result is related to the appearance in Eq. (3) of a single frequency.

When a goes to infinity, a classical "frozen turbulence" scaling is found: D/a goes to the limit C given in Eq. (16). We have examined this expression for power weights $p(k) \propto k^{-\gamma}$. In the limit of a continuous spectrum, C is then readily evaluated as

$$C(\nu,\gamma) = \frac{2}{\pi}\lambda_{\max}\left[\frac{(\gamma-2)}{(\gamma-1)}\frac{(1-\nu^{2(\gamma-1)})}{(1-\nu^{2(\gamma-2)})}\right]^{1/2}, \tag{80}$$

where $\nu = k_{\min}/k_{\max}$. The limit of this expression for $\nu \to 1$ coincides with the one-wave diffusion model:

$$C(1,\gamma) = \frac{2}{\pi}\lambda_0 \equiv \frac{4}{k_0}. \tag{81}$$

[This is obtained at once from Eq. (16); using (9) to (11), it yields (33).] On the other hand, when ν goes from 1 to 0, C decreases towards

$$C(0,\gamma) = C(1,\gamma)\left[\frac{\gamma-2}{\gamma-1}\right]^{1/2}. \tag{82}$$

Direct evaluations of $C(\nu,\gamma)$ allow us to see that (81) and (82) are in fact upper and lower bounds for the ratio D/a for isotropic spectra. Therefore, taking account of the detailed structure of the wave number spectrum does not bring the theoretical description closer to the simulation results; for example, for $\gamma = 3$ (the value chosen in Ref. [7]), the bounds for D/a are only separated by a factor $\sqrt{2}$ and the discrete spectrum corrections displace the theoretical values (very slightly) further away from the simulation points (see Fig. 1).

We have noted in Sec. I that, since D/a and D/a^2 have the dimensions of a length and a time, respectively, their asymptotic limits [Eqs. (15) to (17)] allow us to define a model-dependent system of units, similar to that introduced in Ref. [7] and used in Fig. 1, which makes all isotropic models obey the same asymptotic laws, of course. The few numerical results above show that this is more than a dimensional argument. In particular, as Fig. 1 indicates, when $D(a)$ is computed with these units, all curves (for isotropic fields ranging from the one-wave diffusion model to the continuous case with $\nu = 0$) come very close together for *all* values of the turbulence amplitude.

ACKNOWLEDGMENTS

Among the many friends and colleagues who have played a part in this work, the author feels particularly indebted to R. Balescu, M. Baus, T. Erneux, J. H. Misguich, M. Parker, M. Pettini, and J. W. Turner. He also wishes to thank J. Leroy, F. Hermans, E. Kestemont, and J. Orban for their valuable technical help.

APPENDIX: TWO-VARIABLE EXPANSION DERIVATION OF EQS. (45) AND (46)

The small-ϵ asymptotic solution of Eq. (28), obtained in Sec. III B, can be arrived at using the two-variable expansion procedure [13]. To sketch this derivation, we introduce the variables $x_0 = x$ and $x_1 = \epsilon x$ and expand $f(x_0, x_1, \epsilon)$ in a perturbation series:

$$f(x_0,x_1,\epsilon) = f^{(0)}(x_0,x_1) + \epsilon f^{(1)}(x_0,x_1) + \epsilon^2 f^{(2)}(x_0,x_1) + \cdots . \tag{A1}$$

The expansion of the right-hand side then yields

$$e^{-f} = e^{-f^{(0)}}(1 - \epsilon f^{(1)} + \cdots) \tag{A2}$$

and, with the chain rule for the derivatives, (28) becomes

$$\left(\frac{\partial^2}{\partial x_0^2} + 2\epsilon\frac{\partial^2}{\partial x_0 \partial x_1} + \epsilon^2\frac{\partial^2}{\partial x_1^2}\right)(f^{(0)} + \epsilon f^{(1)} + \cdots) = \epsilon\cos(x_0)\, e^{-f^{(0)}}(1 - \epsilon f^{(1)} + \cdots). \tag{A3}$$

The initial conditions for the $f^{(n)}$ are obtained by expanding those for Eq. (28).

The order zero equation is simply

$$\frac{\partial^2 f^{(0)}}{\partial x_0^2} = 0; \tag{A4}$$

therefore,

$$f^{(0)}(x_0,x_1) = \psi^{(0)}(x_1) + x_0\widetilde{\psi}^{(0)}(x_1), \tag{A5}$$

with the initial conditions

$$f^{(0)}(0,0) \equiv \psi^{(0)}(0) = 0, \tag{A6a}$$

$$\left.\frac{\partial f^{(0)}}{\partial x_0}\right|_{0,0} \equiv \widetilde{\psi}^{(0)}(0) = 0. \tag{A6b}$$

The first-order equation is

$$\frac{\partial^2 f^{(1)}}{\partial x_0^2} + 2\frac{\partial^2 f^{(0)}}{\partial x_0 \partial x_1} = \cos(x_0)\, e^{-f^{(0)}}. \tag{A7}$$

The second term in this equation equals $2d\widetilde{\psi}^{(0)}(x_1)/dx_1$, does not depend on x_0 and would therefore make a contribution $\propto (x_0)^2$ to $f^{(1)}$, which would dominate $f^{(0)}$ for large values of x_0; hence, this term is secular and must vanish. This then says that $\widetilde{\psi}^{(0)}$ does not depend on x_1 and by (A6b) is equal to zero; thus,

$$f^{(0)} \equiv \psi^{(0)}(x_1) \tag{A8}$$

does not depend on x_0. Equation (A7) then simplifies to the following linear inhomogeneous form:

$$\frac{\partial^2 f^{(1)}}{\partial x_0^2} = \cos(x_0)\, e^{-\psi^{(0)}(x_1)}. \tag{A9}$$

The solution of (A9) (and of the higher-order equations) is the sum of the general solution of the homogeneous equation [as in (A5), without the secular term $\propto x_0$] plus a particular solution obtained by direct integration; thus,

$$f^{(1)} = \psi^{(1)}(x_1) - \cos(x_0)\, e^{-\psi^{(0)}(x_1)}, \tag{A10}$$

with the initial conditions

$$f^{(1)}(0,0) = 0 \longrightarrow \psi^{(1)}(0) = 1, \tag{A11a}$$

$$\left.\frac{\partial f^{(1)}}{\partial x_0}\right|_{0,0} + \left.\frac{\partial f^{(0)}}{\partial x_1}\right|_{0,0} = 0 \longrightarrow \left.\frac{d\psi^{(0)}}{dx_1}\right|_{0} = 0. \tag{A11b}$$

Equations (A8) and (A10) confirm our solution [Eq. (45)].

To determine $\psi^{(0)}(x_1)$, we must go to order ϵ^2:

$$\frac{\partial^2 f^{(2)}}{\partial x_0^2} + 2\frac{\partial^2 f^{(1)}}{\partial x_0 \partial x_1} + \frac{\partial^2 f^{(0)}}{\partial x_1^2} = \cos(x_0)\, e^{-f^{(0)}}(-f^{(1)}) \tag{A12}$$

or

$$\frac{\partial^2 f^{(2)}}{\partial x_0^2} + 2\sin(x_0)\frac{de^{-\psi^{(0)}(x_1)}}{dx_1} + \frac{d^2\psi^{(0)}}{dx_1^2} = -\cos(x_0)\psi^{(1)}(x_1)\, e^{-\psi^{(0)}(x_1)} + \frac{1}{2}[1+\cos(2x_0)]\, e^{-2\psi^{(0)}(x_1)}. \tag{A13}$$

In addition to contributions that oscillate (around a zero average), this equation involves two secular terms (independent of x_0), the canceling of which leads to Eq. (46) for $\psi^{(0)}$.

[1] P. C. Liewer, Nucl. Fusion **25**, 543 (1985).

[2] D. Montgomery, in *Plasma Physics: Les Houches 1972*, edited by C. Dewitt and J. Peyraud (Gordon and Breach, New York, 1975); Physica C **82**, 111 (1976).

[3] J. B. Taylor and B. McNamara, Phys. Fluids **14**, 1492 (1971); see also J. Weinstock, *ibid.* **11**, 354 (1968).

[4] T. H. Dupree, Phys. Fluids **9**, 1773 (1966); **10**, 1049 (1967).

[5] J. H. Misguich, R. Balescu, H. L. Pecseli, T. Mikkelsen, S. E. Larsen, and Qiu Xiaoming, Plasma Phys. Controlled Fusion **29**, 825 (1987).

[6] M. Pettini, A. Vulpiani, J. H. Misguich, R. Balescu, M. De Leener, and J. Orban, Commissariat à l'Energie Atomique EUR-CEA-FC Report No. 1271, 1986 (unpublished).

[7] M. Pettini, A. Vulpiani, J. H. Misguich, M. De Leener, J. Orban, and R. Balescu, Phys. Rev. A **38**, 344 (1988).

[8] J. H. Misguich and R. Nakach, Phys. Rev. A **44**, 3869 (1991); for a recent review on Lagrangian chaos, see for instance, A. Crisanti, M. Falcioni, G. Paladin, and A. Vulpiani, Riv. Nuovo Cimento **14**, No. 12 (1991).

[9] M. De Leener, J. H. Misguich and R. Balescu, in *Proceedings of the Second European Fusion Theory Meeting, Varenna 1987* (ISPP, Milano, 1987), p. 14.

[10] S. Corssin, in *Atmospheric Diffusion and Air Pollution*, Advances in Geophysics Vol. 6, edited by F. N. Frenkiel and P. A. Sheppard (Academic Press, New York, 1959).

[11] G. I. Taylor, Proc. R. Soc. London Ser. A **164**, 476 (1938).

[12] B. B. Kadomtsev, *Plasma Turbulence* (Academic Press, London, 1965); R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969).

[13] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics* (Springer-Verlag, New York, 1981); C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).

[14] As a two-dimensional partial differential problem, this equation was solved by J. Liouville: Liouville's Journal, XVIII, 71 (1853); see H. Bateman, *Partial Differential Equations of Mathematical Physics* (Dover, New York, 1944).